Propagation Modes, Equivalent Circuits, and Characteristic Terminations for Multiconductor Transmission Lines with Inhomogeneous Dielectrics

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Abstract—**The theory of wave propagation on lossless multiconductor transmission lines with inhomogeneous dielectrics is developed using matrix analysis. The treatment is concise and complete and has the advantage of identifying propagation modes in a way that permits straightforward physical interpretation. The equivalent circuit for the general line is derived and its application to the solution of wave problems with reflections is demonstrated. Special consideration is given to the problem of characteristically terminating a multiconductor line, i.e., terminating without reflections. The realizability of such a characteristic termination network is discussed, and proofs of realizability are given for the important cases of all lines with homogeneous dielectrics and all three-conductor lines, regardless of dielectric inhomogeneities. Symmetric three-conductor lines are discussed to exemplify the general theory, and an application to the problem of mode conversion on symmetric and asymmetric shielded strip lines is given.**

I. Introduction

HIGH-FREQUENCY transmission or instrumentation systems using coaxial cables have become a standard feature in nearly every research and development laboratory. Such concepts as characteristic termination, line impedance, propagation velocity, and reflection coefficient in two-conductor systems are familiar to engineers and technicians alike. However, the extension of these concepts to multiconductor systems and their proper application in the design of instrumentation and transmission equipment has not been adequately recognized. For instance, it is important to realize that mode conversion in shielded-pair instrumentation cables (see Fig. 1) will distort the signal impressed between the two principal conductors if the termination impedances between the shield and the conductors are not properly chosen. Besides the application of multiple-line theory to such instrumentation cables, other important applications occur in the cases of microwave directional couplers and other stripline devices.

The primary purposes of this paper are twofold. First, the theory of wave propagation on lossless multiconductor transmission lines with inhomogeneous dielectrics is presented in a way that is completely general and yet concise. While many individual parts of this problem appear elsewhere [1]–[8], the matrix formalism used allows the notation to be kept very compact, with many of the relationships between voltage and current waves and their reflections taking precisely the same form as the analogous equations for the familiar two-conductor line. Unique features of this work are the use of the

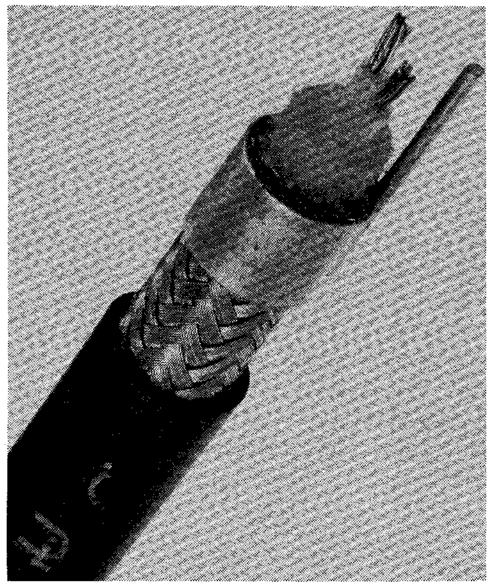

Fig. 1. Shielded-pair instrumentation cable—an example of a three-conductor transmission line.

pair of adjoint matrices[1] $\boldsymbol{LC}$ and $\boldsymbol{CL}$ in generalizing the inhomogeneous dielectric case, and the simple interpretation of the propagation modes as actual voltage and current components.

The second objective of the paper is to derive equivalent circuits and characteristic terminations for multiconductor lines. The equivalent circuit permits considerable simplification in the analysis of wave-propagation problems on such lines. The realization of a characteristic termination network is of interest because it provides the means for eliminating reflected waves.

Derivation of these circuits and terminations is based on the derived relation

$$\boldsymbol{V}(z, t) = \boldsymbol{Z}_o \boldsymbol{I}(z, t)$$

for unidirectional voltage $\boldsymbol{V}$ and current $\boldsymbol{I}$ waves on the line, formally equivalent to the familiar result for two-conductor lines. $\boldsymbol{Z}_o$ is a matrix called the characteristic impedance ma-

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[1] Bold type is used to denote both vectors and matrices. The context should make clear which is intended. Subscripted symbols in bold type refer to specific vectors or matrices (e.g., the ith voltage eigenvector is $\boldsymbol{V}_i$). Symbols for vectors or matrices that appear in regular type, subscripted, refer to components of the vector or matrix (e.g., the 1st component of the ith voltage eigenvector is V_{1i}).

trix. The characteristic termination is, not surprisingly, any network with impedance matrix $\boldsymbol{Z}_o$. The physical realizability of such a termination in terms of a resistive network is discussed. Rigorous proofs of realizability are given for the important cases of all lines with homogeneous dielectrics and all three-conductor lines in general, regardless of dielectric inhomogeneities.

Finally, the theory is applied to a treatment of some general aspects of symmetric three-conductor lines. Some illustrative examples of mode conversion on shielded striplines are given.

II. Derivation of the Propagation Modes

The propagation modes of a general transmission line are derived as follows. Consider a lossless line formed by $n+1$ conductors, one of which is chosen as the reference ground. The line is assumed to be uniform along its length (the z-coordinate), but of arbitrary cross-sectional configuration. In particular, the dielectric material may be inhomogeneous; this freedom requires special consideration in the analysis.

In the presence of materials of different dielectric constants, the propagation cannot in general be TEM. However, the low-frequency propagation is "quasi-TEM" [8], [9], and a valid analysis of the propagation modes can proceed from the telegrapher's equations. These equations are [1]–[3], [9], [10]:

$$\frac{\partial \boldsymbol{V}(z, t)}{\partial z} = -\boldsymbol{L}\frac{\partial \boldsymbol{I}(z, t)}{\partial t} \tag{1}$$

$$\frac{\partial \boldsymbol{I}(z, t)}{\partial z} = -\boldsymbol{C}\frac{\partial \boldsymbol{V}(z, t)}{\partial t} \tag{2}$$

where $\boldsymbol{V}(z, t)$ and $\boldsymbol{I}(z, t)$ are n-dimensional column vectors which represent the voltages and currents on the conductors, and $\boldsymbol{L}$ and $\boldsymbol{C}$ are $n\times n$ inductance and capacitance (per unit length) matrices. Both $\boldsymbol{L}$ and $\boldsymbol{C}$ are real, symmetric, and dominant. They are, therefore, positive definite [2]. $\boldsymbol{L}$ has all positive elements, and $\boldsymbol{C}$ has all positive diagonal elements and all negative off-diagonal elements.

A propagation mode is defined as a solution to (1) and (2) in which the voltage and current vectors have the following form:

$$\boldsymbol{V}(z, t) = \boldsymbol{V}\cdot f(z - vt)$$

$$\boldsymbol{I}(z, t) = \boldsymbol{I}\cdot f(z - vt).$$

Substitution in (1) and (2) results in the following relations between the constant vectors $\boldsymbol{V}$ and $\boldsymbol{I}$:

$$\boldsymbol{V} = v\boldsymbol{L}\boldsymbol{I} \tag{3}$$

$$\boldsymbol{I} = v\boldsymbol{C}\boldsymbol{V}. \tag{4}$$

Eliminating $\boldsymbol{I}$, there results an eigenvalue equation for $\boldsymbol{V}$:

$$(\boldsymbol{L}\boldsymbol{C})\boldsymbol{V} = \frac{1}{v^2}\boldsymbol{V} \tag{5}$$

that is, $1/v^2$ must be an eigenvalue of the matrix $\boldsymbol{LC}$, and $\boldsymbol{V}$ the associated eigenvector.

For inhomogeneous dielectrics there will in general be n distinct eigenvalues, although degeneracies may occur among the eigenvalues because of symmetry. For homogeneous dielectrics, $\boldsymbol{LC} = (1/v_o^2)\boldsymbol{U}$, where $v_o = 1/\sqrt{\mu\epsilon}$, and $\boldsymbol{U}$ is the identity matrix [2]. Any vector $\boldsymbol{V}$ will then satisfy (5); the eigenvalues are n-fold degenerate, with propagation velocity v_o for any signal. In any case, there will always be n linearly independent vectors available to form a basis in n-dimensional space.

Associated with the eigenvectors $\boldsymbol{V}_i$ and eigenvalues $1/v_i^2$, $i = 1, \cdots, n$, there are current eigenvectors $\boldsymbol{I}_i$ related to the $\boldsymbol{V}_i$ through (4). The $\boldsymbol{I}_i$ are easily shown to be eigenvectors of the adjoint matrix $\boldsymbol{CL}$ with the same eigenvalues $1/v_i^2$.

III. Properties of the Modes

A. *Real and Positive Eigenvalues, Real Eigenvectors*

In order that the modes represent unattenuated traveling waves, the velocities must be real, i.e., the eigenvalues $1/v_i^2$ must be real and positive. This is always true for eigenvalues of $\boldsymbol{LC}$ if $\boldsymbol{L}$ and $\boldsymbol{C}$ are realizable; the proof is briefly outlined as follows. The matrix $\boldsymbol{LC}$ can be written[2]

$$\boldsymbol{LC} = \boldsymbol{L}^{1/2}(\boldsymbol{C}^{1/2}\boldsymbol{L}^{1/2})^T(\boldsymbol{C}^{1/2}\boldsymbol{L}^{1/2})\boldsymbol{L}^{-1/2}.$$

The matrix $\boldsymbol{B} = (\boldsymbol{C}^{1/2}\boldsymbol{L}^{1/2})^T(\boldsymbol{C}^{1/2}\boldsymbol{L}^{1/2})$ has the form $\boldsymbol{A}^T\boldsymbol{A}$, where $\boldsymbol{A}$ is real; hence, $\boldsymbol{B}$ is positive definite [11, pp. 39–40] and has positive eigenvalues. But $\boldsymbol{LC}$ is just a similarity transformation of $\boldsymbol{B}$, so its eigenvalues are the same as those of $\boldsymbol{B}$ [11, p. 48]. Since $\boldsymbol{LC}$ and its eigenvalues are real, the eigenvectors can be assumed real as well.

By taking the v_i to be the positive (negative) square roots of the inverses of the eigenvalues, n propagation modes are established, which have the form of waves traveling in the positive (negative) z direction. Equations (3) and (4) show that for a given voltage (current) eigenvector, the signs of the corresponding currents (voltages) are reversed when the direction of propagation is reversed. This is required from the symmetry of the physical configuration.

B. *Orthogonality Properties*

The orthogonality properties of the modes are exhibited as follows. Consider the ith and jth modes. Equation (5) and the analogous equation for the current eigenvectors yield

$$\boldsymbol{I}_j\cdot\boldsymbol{LC}\boldsymbol{V}_i - \boldsymbol{V}_i\cdot\boldsymbol{CL}\boldsymbol{I}_j = \left(\frac{1}{v_i^2} - \frac{1}{v_j^2}\right)\boldsymbol{I}_j\cdot\boldsymbol{V}_i$$

where the dot denotes inner product. Since $\boldsymbol{CL}$ is the transpose of $\boldsymbol{LC}$ (more generally, the adjoint, since the matrices are real), the left side of this equation is zero. Hence

$$\boldsymbol{I}_j\cdot\boldsymbol{V}_i = 0 \tag{6}$$

unless $v_i = v_j$.

In case of degeneracies, i.e., $v_i = v_j$ for some i and j, there is arbitrariness in the choice of eigenvectors. However, the n linearly independent eigenvectors can always be orthogonalized by a generalization of the Gram–Schmidt procedure.

[2] See [2] for the appropriate definition of the square root of a matrix. The matrices $\boldsymbol{L}^{1/2}$ and $\boldsymbol{C}^{1/2}$ are real and symmetric, and $\boldsymbol{L}^{1/2}\boldsymbol{L}^{1/2} = \boldsymbol{L}$, $\boldsymbol{C}^{1/2}\boldsymbol{C}^{1/2} = \boldsymbol{C}$.

C. Eigenvector Expansions

Now consider only positive v_i, and assume the eigenvectors to be normalized, i.e.

$$I_i \cdot V_j = \delta_{ij}.$$

[That this is possible follows from (6) and the fact that $I_i \cdot V_i$ is always nonzero. Mathematically, this establishes the linear independence in n-space of both sets of eigenvectors V_i, I_i, $i=1, 2, \cdots, n$.] Let M_V and M_I be the matrices whose columns are the normalized voltage and current eigenvectors. Then

$$M_V M_I^T = M_I M_V^T = U.$$

Hence, an arbitrary vector E can be represented as a sum of voltage eigenvectors in the form

$$E = M_V A = \sum_i A_i V_i$$

where A is the vector

$$A = M_I^T E$$

i.e.

$$A_i = I_i \cdot E.$$

Clearly, any vector can be represented in terms of current eigenvectors by an analogous development.

IV. The Characteristic Admittance and Impedance Matrices

The eigenvector expansions will now be used to derive relations between the voltages and currents in waves traveling in either direction. Let a wave traveling in the forward (positive z) direction be characterized at some point in space and time by the voltage vector V_f.

If

$$A = M_I^T V_f$$

then

$$V_f = \sum_i A_i V_i.$$

So the current in this forward wave is

$$I_f = \sum_i A_i I_i = M_I A = (M_I M_I^T V_f) = Y_o V_f \tag{7}$$

where the characteristic admittance matrix Y_o, defined by

$$Y_o = M_I M_I^T \tag{8a}$$

has been introduced. Since $M_I = M_V^{-1}$, it can also be written

$$Y_o = M_I M_V^{-1}. \tag{8b}$$

This latter form is important because it holds regardless of whether the eigenvectors are normalized. All that is required for its validity is that the eigenvectors satisfy (3) and (4).

The characteristic impedance matrix is defined simply as the inverse of this admittance matrix:

$$Z_o = Y_o^{-1}.$$

Then

$$V_f = Z_o I_f. \tag{9}$$

(The next section addresses the question of whether the matrix Y_o is that of a physically realizable resistive network. If so, its inverse Z_o always exists.)

Similarly, for a wave traveling in the backward direction

$$I_b = -Y_o V_b \tag{10}$$

or

$$V_b = -Z_o I_b. \tag{11}$$

V. Realizability of the Admittance Matrix as a Resistive Network

The realizability of the characteristic admittance matrix Y_o will now be considered in order to pave the way for subsequent discussions.

In order to be physically realizable in terms of a resistive network, the matrix Y_o must have the following properties [12]. It must be real, symmetric, dominant (therefore positive definite), and have positive diagonal elements and negative off-diagonal elements. Since M_I is real, (8a) immediately establishes that Y_o is always real, symmetric, and has positive diagonal elements. Furthermore, it is positive definite, since it has the form $A^T A$, with A real [11, pp. 39–40]. Based on physical intuition, it is tempting to conjecture that Y_o is always realizable. However, we are able to give rigorous proofs only for two special cases. Nonetheless, they are important ones; namely, lines with homogeneous dielectrics and three-conductor lines.

To establish the proof for lines with homogeneous dielectrics, first use (4) to write M_I in the form

$$M_I = C M_V S \tag{12}$$

where S is a diagonal matrix whose elements are the propagation velocities v_i. Then

$$Y_o = C M_V S M_V^{-1}. \tag{8c}$$

If the dielectric is homogeneous, $S = (1/\sqrt{\mu\epsilon})\ U = v_o U$, then

$$Y_o = v_o C.$$

Since the matrix C has precisely the same properties required for realizability of Y_o, it is established that Y_o is always realizable for a line with a homogeneous dielectric.

The proof for general three-conductor lines is as follows. Use (3) to write

$$M_V = L M_I S. \tag{13}$$

From (8c) and (13)

$$Y_o = C M_V S S^{-1} M_I^{-1} L^{-1} = C Y_o^{-1} L^{-1}$$

so

$$Y_o L = C Y_o^{-1} \tag{14a}$$

or

$$Y_o L Y_o = C. \tag{14b}$$

This is a general result, valid for any line.

For a three-wire line, these matrices can be written

$$L = \begin{pmatrix} L_1 & L_m \\ L_m & L_2 \end{pmatrix} \qquad C = \begin{pmatrix} C_1 & -C_m \\ -C_m & C_2 \end{pmatrix}$$

$$Y = \begin{pmatrix} Y_1 & Y_m \\ Y_m & Y_2 \end{pmatrix} \tag{15}$$

where L_1, L_2, L_m, C_1, C_2, and C_m are positive and $L_m < L_1$, $L_m < L_2$, $C_m < C_1$, $C_m < C_2$. It has been established that Y_1 and Y_2 are positive, so (14b) implies that Y_m is negative, as required. To show dominance write out the diagonal components of the matrix equation (14a):

$$Y_1 L_m + Y_m L_2 = -\frac{1}{\Delta}(C_1 Y_m + C_m Y_1)$$

$$Y_m L_1 + Y_2 L_m = -\frac{1}{\Delta}(C_m Y_2 + C_2 Y_m)$$

where $\Delta = Y_1 Y_2 - Y_m^2$. Since $\boldsymbol{Y}_o$ is known to be positive definite, $\Delta > 0$.[3]

Then

$$\frac{-Y_m}{Y_1} = \frac{(L_m + C_m/\Delta)}{(L_2 + C_1/\Delta)} < 1$$

$$\frac{-Y_m}{Y_2} = \frac{(L_m + C_m/\Delta)}{(L_1 + C_2/\Delta)} < 1.$$

Hence, $\boldsymbol{Y}_o$ is dominant. Realizability of $\boldsymbol{Y}_o$ for any three-wire line is therefore established.

VI. The Equivalent Circuit for the Multiconductor Line

Consider a line of length l, connected to arbitrary $(n+1)$-terminal networks at each end, as shown in Fig. 2. At the end $z = l$, the voltage and current vectors at any time can be written as the sum of forward and backward vector waves:

$$\boldsymbol{V}_l = \boldsymbol{V}_{fl} + \boldsymbol{V}_{bl}$$

$$\boldsymbol{I}_l = \boldsymbol{I}_{fl} + \boldsymbol{I}_{bl}. \tag{16}$$

Because the forward and backward waves satisfy (9) and (11), respectively, it is possible to eliminate the backward wave from these equations to obtain

$$\boldsymbol{V}_l + \boldsymbol{Z}_o \boldsymbol{I}_l = 2\boldsymbol{V}_{fl} \tag{17}$$

a form familiar from the theory of two-conductor lines. If the incident wave $\boldsymbol{V}_{fl}$ is known, then (17) and a knowledge of the terminating network suffice to determine $\boldsymbol{V}_l$ and $\boldsymbol{I}_l$. Similarly, at the end $z = 0$, the voltage and current obey

$$\boldsymbol{V}_o - \boldsymbol{Z}_o \boldsymbol{I}_o = 2\boldsymbol{V}_{bo} \tag{18}$$

where $\boldsymbol{V}_{bo}$ is the backward (incident) wave at $z = 0$.

Equations (17) and (18) are just the equations for the response of the circuit shown in Fig. 3. Hence this circuit is an equivalent circuit for the multiconductor line. Each end of the line responds as an $(n+1)$-terminal network with impedance matrix $\boldsymbol{Z}_o$ (admittance matrix $\boldsymbol{Y}_o$), with each terminal connected in series with a voltage source. The voltage sources are just twice the components of the appropriate incident voltage vectors $\boldsymbol{V}_{fl}$ and $\boldsymbol{V}_{bo}$.

[3] The principal minors of a positive definite matrix are positive [13, p. 258].

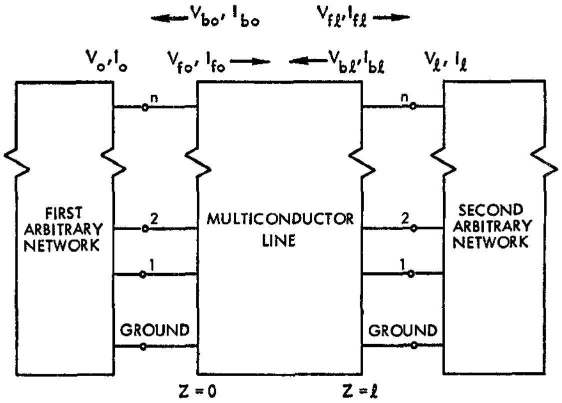

Fig. 2. Schematic of multiconductor line connected to networks at each end. Pairs of voltage and current vectors are shown with arrows to indicate the directions of propagation of the vectors referred to in the text.

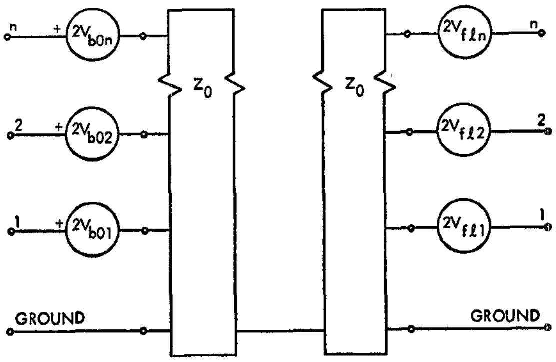

Fig. 3. Equivalent circuit for the multiconductor line.

To determine these sources, use (16) and its equivalent at $z = 0$ to obtain

$$\boldsymbol{V}_{bl} = \tfrac{1}{2}(\boldsymbol{V}_l - \boldsymbol{Z}_o \boldsymbol{I}_l)$$

$$\boldsymbol{V}_{fo} = \tfrac{1}{2}(\boldsymbol{V}_o + \boldsymbol{Z}_o \boldsymbol{I}_o).$$

Since, in general, the different modes propagate at different velocities, a knowledge of $\boldsymbol{V}_{fo}$ at one time does not suffice to identify $\boldsymbol{V}_l$ one "transit time" later. Instead, $\boldsymbol{V}_{fo}$ must be decomposed into eigenvectors at $z = 0$:

$$\boldsymbol{V}_{fo}(t) = \boldsymbol{M}_V \boldsymbol{A}(t)$$

where

$$\boldsymbol{A}(t) = \boldsymbol{M}_I^T \boldsymbol{V}_{fo}(t)$$

$$= \boldsymbol{M}_I^T [\boldsymbol{V}_o(t) + \boldsymbol{Z}_o \boldsymbol{I}_o(t)]/2.$$

Then define the transit time for each mode as

$$\tau_i = l/v_i, \qquad i = 1, 2, \cdots, n.$$

The desired voltage vector $\boldsymbol{V}_{fl}(t)$ is thus obtained from $\boldsymbol{V}_{fo}(t)$ by adding eigenvectors at the appropriate transit time after leaving the point $z = 0$:

$$\boldsymbol{V}_{fl}(t) = 1/2 \sum_j A_j(t - \tau_j) \boldsymbol{V}_j$$

$$= 1/2 \sum_j \{\boldsymbol{M}_I^T [\boldsymbol{V}_o(t - \tau_j) + \boldsymbol{Z}_o \boldsymbol{I}_o(t - \tau_j)]\}_j \boldsymbol{V}_j$$

where the subscript j on the { } indicates the jth component of the enclosed vector.

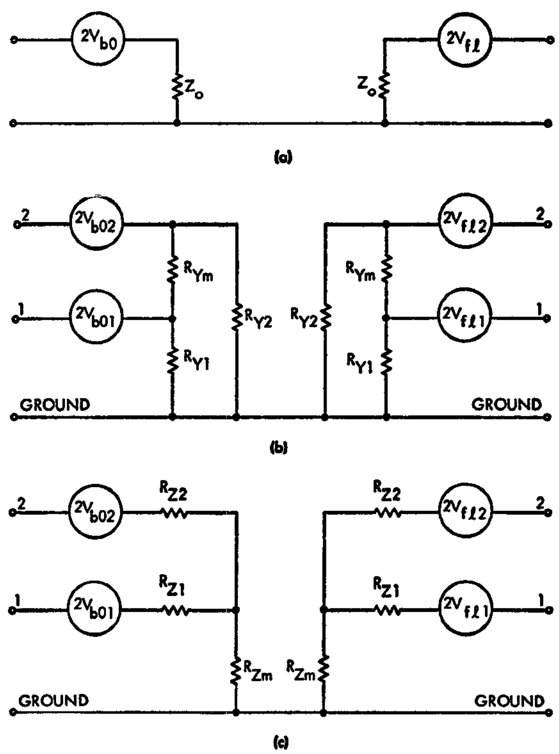

Fig. 4. Equivalent circuits for two- and three-conductor lines. (a) Two-wire line where Z_o is the characteristic impedance. (b) One possible circuit for a three-wire line. (c) A second possible circuit for a three-wire line.

For the cases of homogeneous dielectrics (and two-conductor lines), for which there is a unique transit time τ, this becomes simply

$$V_{fl}(t) = 1/2[V_o(t-\tau) + Z_oI_o(t-\tau)].$$

Similarly, at $z=0$

$$V_{bo}(t) = 1/2 \sum_j \{M_I[V_l(t-\tau_j) - Z_oI_l(t-\tau_j)]\}_j V_j$$

and for homogeneous dielectrics,

$$V_{bo}(t) = 1/2[V_l(t-\tau) - Z_oI_l(t-\tau)].$$

Equivalent circuits for two- and three-conductor lines are shown in Fig. 4.[4] The circuit for the two-conductor line is well known, and is given only to demonstrate that it falls within the framework of the general theory.

The equivalent circuit for any multiconductor line has some arbitrariness in that the impedance network can be represented in a variety of ways, provided that it has impedance matrix Z_o. Fig. 4(b) and (c) shows two convenient forms for a three-conductor line. The first is labeled with subscript Y's because the resistance values are more simply related to the components of the matrix Y_o than to those of Z_o. Adopting the notation of (15), the relationships are

$$R_{Y1} = \frac{1}{Y_1 + Y_m}$$

$$R_{Y2} = \frac{1}{Y_2 + Y_m}$$

$$R_{Ym} = \frac{-1}{Y_m}. \tag{19}$$

The resistances in the circuit shown in Fig. 4(c) are more simply related to the components of Z_o. If Z_o is written as

$$Z_o = \begin{pmatrix} Z_1 & Z_m \\ Z_m & Z_2 \end{pmatrix}$$

they are

$$R_{Z1} = Z_1 - Z_m$$

$$R_{Z2} = Z_2 - Z_m$$

$$R_{Zm} = Z_m. \tag{20}$$

The equivalent circuit actually used in an application can be chosen for convenience, depending on the terminating network, type of signals considered, etc.

VII. The Transmission and Reflection Matrices

Using the relationships between waves which were discussed in the previous section, it is a simple matter to derive the matrix equivalents of the familiar transmission and reflection coefficients. The derivation is identical in form to that usually given for transmission and reflection coefficients of two-conductor lines [14].

Assume now that the line shown in Fig. 2 is terminated in a passive circuit with impedance matrix Z_L. (Consider the second network shown in Fig. 2 to be such a circuit.) Consider a voltage signal V_{fl} incident on this termination. As in (16) *et seq.*, let the voltage and current vectors at the terminals be V_l and I_l. They must be related by

$$I_l = Z_L^{-1}V_l.$$

Then from (17)

$$(U + Z_oZ_L^{-1})V_l = 2V_{fl}.$$

Hence the transmitted voltage is given in terms of the incident voltage by

$$V_l = \tau_v V_{fl}$$

where

$$\tau_v = 2Z_L(Z_L + Z_o)^{-1} \tag{21}$$

is the voltage transmission matrix.

From (16), the reflected voltage vector is

$$V_{bl} = V_l - V_{fl}$$
$$= (\tau_v - U)V_{fl}$$
$$= \rho_v V_{fl}$$

where

$$\rho_v = \tau_v - U$$
$$= (Z_L - Z_o)(Z_L + Z_o)^{-1} \tag{22}$$

is the voltage reflection matrix. Note that (21) and (22) are identical in form to those for the usual transmission and reflection coefficients for two-conductor lines [14].

[4] The equivalent circuit shown in Fig. 4(b) was first deduced by Williams and Hull [13]. It was their work that suggested the general results given here.

It is of particular importance to note that the choice $Z_L = Z_o$ yields

$$\tau_v = U$$

$$\rho_v = 0.$$

That is, if the line is terminated in a network which has an impedance matrix equal to the characteristic impedance matrix, then no reflections occur and the output signal is equal to the incident signal. This, of course, is anticipated intuitively on the basis of familiarity with the result for two-conductor lines. It is also immediately apparent upon consideration of the equivalent circuit of Fig. 3. Connecting a network with impedance matrix Z_o to the line is the same as connecting another infinitely long line with identical characteristics, in which case no reflections would occur.

A discussion of the physical realizability of such a termination has already been given in Section V.

VIII. Symmetric Three-Conductor Lines

The concepts which have been developed for general multiconductor lines will now be applied to the specific case of symmetric three-conductor lines.

For any three-conductor line with inductance and capacitance matrices given by (15), the equation $|LC - (1/v^2)U| = 0$ results in the following formula for two propagation velocities:

$$v = \left[\frac{L_1C_1 + L_2C_2 - 2L_mC_m \pm \sqrt{(L_1C_1 - L_2C_2)^2 + 4(L_mC_1 - L_2C_m)(L_mC_2 - L_1C_m)}}{2}\right]^{-1/2}. \tag{23a}$$

If conductors 1 and 2 are symmetric with respect to ground, $L_1 = L_2$ and $C_1 = C_2$, i.e., the inductance and capacitance matrices have the forms

$$L = \begin{pmatrix} L_1 & L_m \\ L_m & L_1 \end{pmatrix} \quad C = \begin{pmatrix} C_1 & -C_m \\ -C_m & C_1 \end{pmatrix}.$$

Then the two velocities defined by (23a) are

$$v_e = 1/\sqrt{(L_1 + L_m)(C_1 - C_m)}$$

$$v_o = 1/\sqrt{(L_1 - L_m)(C_1 + C_m)}. \tag{23b}$$

The voltage and current eigenvectors are

$$V_e = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad I_e = \sqrt{\frac{C_1 - C_m}{L_1 + L_m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{z_e}V_e$$

$$V_o = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad I_o = \sqrt{\frac{C_1 + C_m}{L_1 - L_m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{z_o}V_o.$$

This is a well-known result [2], [15]; the two propagation modes are an even (or common) mode and an odd (or signal) mode. The voltage and current vectors are related by simple scalars, called the even-mode impedance

$$z_e = \sqrt{\frac{L_1 + L_m}{C_1 - C_m}}$$

and the odd-mode impedance,

$$z_o = \sqrt{\frac{L_1 - L_m}{C_1 + C_m}}.$$

It is always true that

$$z_o \leq z_e$$

equality holds only for decoupled lines, for which $L_m = C_m = 0$.

For many applications, it is simpler to leave the eigenvectors unnormalized. Then, the eigenvector matrices

$$M_V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad M_I = \begin{bmatrix} \frac{1}{z_e} & \frac{1}{z_o} \\ \frac{1}{z_e} & \frac{-1}{z_o} \end{bmatrix}$$

can be used.

The characteristic admittance matrix Y_o can be determined from (8b); Z_o is obtained by inversion. The results are as follows:

$$Y_o = \frac{1}{2}\begin{bmatrix} \frac{1}{z_e} + \frac{1}{z_o} & \frac{1}{z_e} - \frac{1}{z_o} \\ \frac{1}{z_e} - \frac{1}{z_o} & \frac{1}{z_e} + \frac{1}{z_o} \end{bmatrix}$$

$$Z_o = \frac{1}{2}\begin{pmatrix} z_e + z_o & z_e - z_o \\ z_e - z_o & z_e + z_o \end{pmatrix}. \tag{24}$$

Consider terminating a three-conductor line with a network of the same form as that which is shown in the equivalent circuit in Fig. 4(b). From (19) and (24) it is found that the values of resistance required for the characteristic termination are

$$R_{Y1} = R_{Y2} = z_e$$

$$R_{Ym} = \frac{2z_e z_o}{z_e - z_o}. \tag{25}$$

Note that R_{Y1} and R_{Y2} are equal to the characteristic even-mode impedance. Analysis of (22) shows that resistances equal to z_e between each conductor and ground will terminate even-mode signals without reflection, independent of the value of mutual resistance. This is, of course, due to the fact that, for an even-mode signal, the two conductors are at the same potential.

Similarly, the combination of resistances that permits termination of odd-mode signals without reflections is somewhat arbitrary. All that is required is that the network be balanced (since $R_{Y1} = R_{Y2}$), and that the impedance seen between terminals 1 and 2 with the ground open be equal to $2z_o$. This is the impedance between conductors 1 and 2 when the line is operated in signal mode. (Sometimes called the signal-mode impedance.)

However, only the choices indicated by (25) will terminate *both* modes. This fact is of importance in terminating shielded-pair cables, such as RG/22, to eliminate noise in the signal mode arising from the conversion of spurious common-mode signals to signal-mode signals upon reflection.

From (20) and (24), the values of R_{Z1}, R_{Z2}, and R_{Zm} required in the circuit of Fig. 4(c) are:

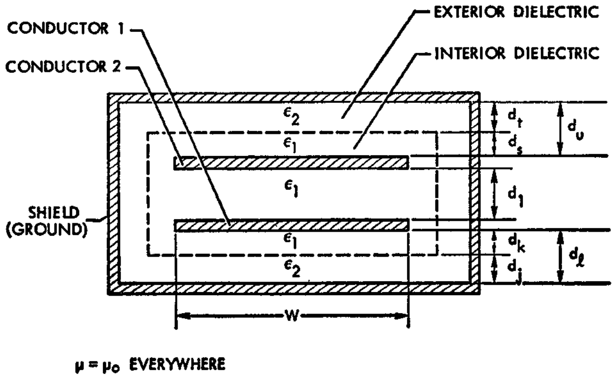

Fig. 5. Cross section of shielded stripline. (Note that the scale belies the assumption $w \gg d = d_l + d_1 + d_u$.)

TABLE I

PARAMETERS CALCULATED FOR THE SYMMETRIC LINE AND TWO ASYMMETRIC LINES

		$d_j = .015''$	$d_j = .01''$	$d_j = 0$	
propagation velocities	v_1	.683c	.673c	.589c	(c = vel. of light in vacuum)
	v_2	.561c	.562c	.540c	
Voltage eigenvector matrix.	M_V	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & .821 \\ 1.020 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -793.856 \\ 3 & -1 \end{pmatrix}$	volts
Characteristic impedance matrix.	Z_o	$\begin{pmatrix} 11.98 & 8.60 \\ 8.60 & 11.98 \end{pmatrix}$	$\begin{pmatrix} 9.91 & 7.11 \\ 7.11 & 10.92 \end{pmatrix}$	$\begin{pmatrix} 3.56 & 2.54 \\ 2.54 & 7.62 \end{pmatrix}$	ohms

Note: The first (second) mode is the one corresponding to the negative (positive) sign of the square root in (23a) for the propagation velocities. Note that unnormalized eigenvectors are used.

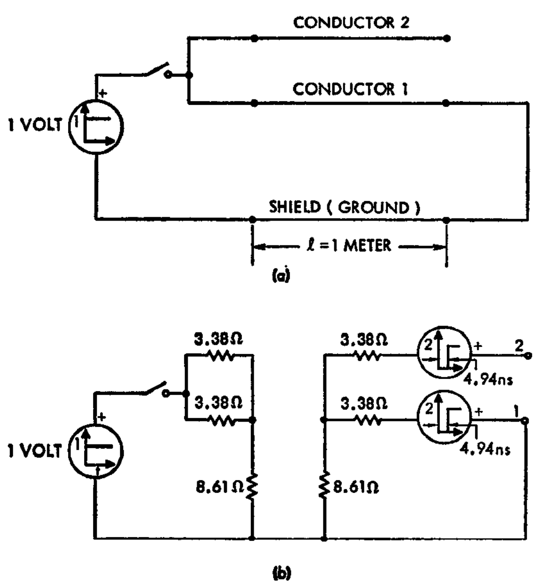

Fig. 6. Circuit illustrating common-mode to signal-mode conversion. (a) Circuit. (b) Equivalent circuit for the case $d_j = 0.015$ m.

$$R_{Z1} = R_{Z2} = z_o$$

$$R_{Zm} = (z_c - z_o)/2. \qquad (26)$$

IX. EXAMPLE: SHIELDED STRIPLINE

Consider the shielded stripline shown in Fig. 5. Under the assumption $W \gg d$ (so fringe fields can be ignored), the inductance and capacitance matrices are easy to evaluate analytically via the usual magnetostatic and electrostatic analysis.

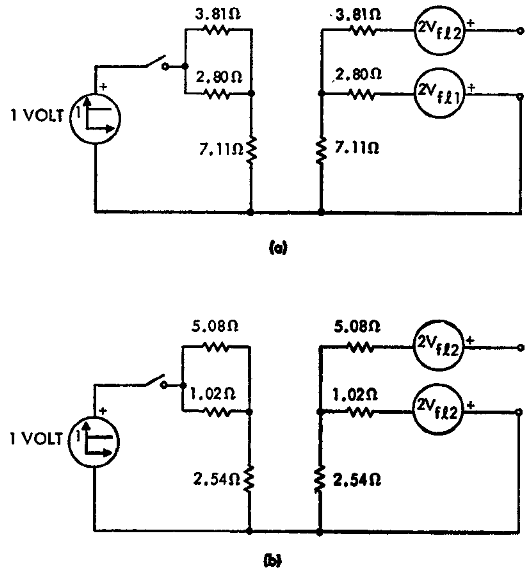

Fig. 7. Equivalent circuits for the asymmetric lines. (a) Circuit for the case $d_j = 0.01$ in. (b) Circuit for the case $d_j = 0$.

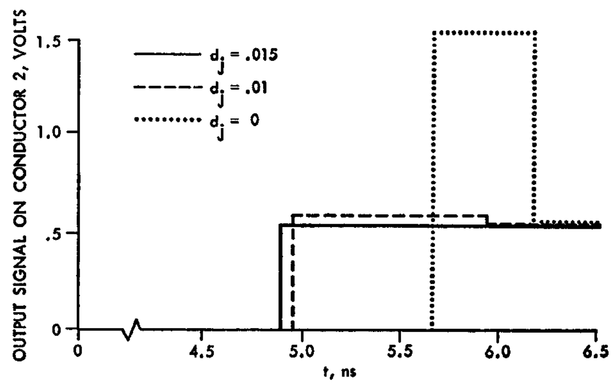

Fig. 8. Output signals versus time for the symmetric and asymmetric lines inserted in the circuit of Fig. 6 (a).

As numerical examples let $\epsilon_1 = 3.43\ \epsilon_0$, $\epsilon_2 = 1.90\ \epsilon_0$, $W = 0.25$ in, $d_1 = 0.01$ in, $d_k = d_s = 0.005$ in, $d_t = 0.015$ in, and consider the three cases $d_j = 0.015$ in (symmetric line), 0.010 in, and 0.0 in. All the pertinent parameters have been computed for these cases and some of them are summarized in Table I.

As an example of mode conversion at an improper termination, consider the circuit shown in Fig. 6(a); it is terminated in a short circuit from conductor 1 to ground. Appropriate equivalent circuits are shown in Figs. 6(b), and 7(a) and (b). (As shown, they are valid only for two transit times, i.e., until reflected waves reach the end $z = 0$.) The unit step input produces a 1-V signal propagating to the right on both conductors. For the symmetric line, this is a pure common-mode signal. However, in the asymmetric lines, both modes are excited so that parts of the signals arrive at the termination at different times. The resulting differential signals between conductors 2 and 1 at the termination are shown in Fig. 8. Note that the difference in transit time results in an overshoot of the signals with respect to their long-time values.

X. CONCLUDING REMARKS

A matrix analysis of lossless multiconductor transmission lines with inhomogeneous dielectrics has been given. The equivalent circuit for such a line has been derived from the analysis. The general theory provides a convenient method for investigating the characteristics of the propagation modes and the impedance properties of the lines. Straightforward techniques for solving problems have been given and some examples have been worked out for the important case of the three-conductor line.

It should be possible to extend the analysis given here to the case of low-loss lines, where exponential attenuation of the propagating modes will arise, along with continuous conversion of energy from one mode to another. This will be the subject of further investigation.

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Excess Losses in *H*-Plane Loaded Waveguides

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***Abstract*—The attenuation in a waveguide partially filled with absorbing material can become larger than that of the same waveguide completely filled with that same material. Theoretical and experimental results are presented together with field distributions showing that this excess loss is due to a large concentration of electric field within the lossy dielectric in the partially filled configuration.**

I. Introduction

IN A RECENT publication, Bui and Gagné [1] determined the attenuation in waveguides containing H-plane slabs of a lossy dielectric, utilizing a perturbation of the lossless dielectric solution. A most interesting feature of the results presented is that, in several configurations involving high-permittivity dielectrics, larger losses were found in partially loaded waveguides than in completely filled ones. Rather surprised by this unexpected result, the authors suggest that it might be attributed to the approximate nature of the technique used. If true, this would mean that the method and the results presented in [1] are not reliable.

The present study shows that, surprising as they may seem at first, the results obtained in [1] correspond to actual fact and that the attenuation is not necessarily a monotonic function of the filling factor. The "excess" attenuation is caused by the presence of a large concentration of the electric field within the dielectric for the partially loaded waveguide. A similar nonmonotonic behavior appears in results previously published by Arnold and Rosenbaum [2].

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II. Theoretical Results

Since a number of publications have already dealt in some detail with this type of structure [3]–[5], there is no need to repeat the basic theory here. The complex transcendental equation obtained for lossy-dielectric loading can be solved exactly by means of available computer programs [6]. Calculations were made for the longitudinal section magnetic LSM_{11} mode in a waveguide containing a lossy slab next to the broad wall (Fig. 1). Results for the attenuation and phase shift are presented in Figs. 2 and 3 as a function of slab thickness for different conductivities. For conductivities σ much smaller than $\omega\epsilon$, the attenuation curves increase exponentially at first then pass through a maximum in the vicinity of $t/a=0.24$ (for this particular configuration), and finally taper down to the value for the completely filled guide.

For large conductivities [Fig. 2(b)], the attenuation curves behave differently. The attenuation increases sharply for thin slabs (as in the previous case), but the peak of the curve is